

Approximations for Gibbs states of arbitrary Hölder potentials on hyperbolic folded sets

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Abstract

In the case of smooth non-invertible maps which are hyperbolic on folded basic sets Λ , we give approximations for the Gibbs states (equilibrium measures) of arbitrary Hölder potentials, with the help of weighted sums of atomic measures on preimage sets of high order. Our endomorphism may have also stable directions on Λ , thus it is non-expanding in general. Folding of the phase space means that we do not have a foliation structure for the local unstable manifolds (they depend on the whole past and may intersect each other both inside and outside Λ), and moreover the number of preimages remaining in Λ may vary; also Markov partitions do not always exist on Λ . Our convergence results apply also to Anosov endomorphisms, in particular to Anosov maps on infranilmanifolds.

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1 Introduction

Gibbs states (equilibrium measures) of Hölder potentials for smooth maps appear naturally in statistical mechanics (for example [2], [15], [16], etc). In order to study equilibrium measures for smooth diffeomorphisms, one can use the specification property in order to find plenty of periodic points, which in turn can be used to approximate equilibrium measures. This method was employed successfully by Bowen in [2] (see also [3]). Also in [2] it is studied the weighted distribution of preimages for one-sided shifts of finite type (which are examples of non-invertible maps).

The setting that we work with in this paper is *different* from both the hyperbolic diffeomorphism case (see [2]), as well as from the expanding case (see [13]). We assume that $f : M \rightarrow M$ is a smooth map (endomorphism), not necessarily a diffeomorphism, and that Λ is a basic set for f so that f is hyperbolic on Λ , but not necessarily expanding. For such *non-expanding endomorphisms* there are no Markov partitions in general, so it is not possible to code the system using Markov partitions like in the diffeomorphism case. Also the presence of stable directions on Λ makes the local inverse iterates of small balls to grow exponentially (up to a certain order). For instance in [1] it was introduced a family of horseshoes with self intersections, and it was proved that there are open

sets of parameters which give non-injectivity of the map on the respective basic set. In the non-invertible case, if f is hyperbolic on Λ , we do not have a foliation structure for the local unstable manifolds; the local unstable manifolds depend now on the whole prehistories (see [14]). This folding of the phase space is a major difference from the diffeomorphism case, since we are forced to work on $\hat{\Lambda}$ which is not a manifold. The local unstable manifolds may intersect each other and through any given point there may pass infinitely many local unstable manifolds. Moreover the number of f -preimages belonging to Λ *may vary* from point to point, so the map is not necessarily constant-to-1; the set Λ is **not** necessarily totally invariant. The local unstable manifolds depend Holder on their respective prehistories in the canonical metric on $\hat{\Lambda}$ ([8]). The existence of several n -preimages in Λ for any point $x \in \Lambda$ means that we can have n -preimages $y \in f^{-n}(x) \cap \Lambda$ of x where the consecutive sum $S_n\phi(y) := \phi(y) + \dots + \phi(f^{n-1}y)$ is *well behaved*, but also other n -preimages $z \in f^{-n}(x) \cap \Lambda$ where $S_n\phi(z)$ is *badly behaved*.

In the case of smooth non-invertible expanding maps the situation is difficult, and the problem of finding the weighted distributions of preimages was solved by Ruelle in [13]; in that situation it was important that the local inverse iterates contract uniformly on small balls.

In our present non-invertible non-expanding setting, we will describe the weighted distributions of preimages by studying the intersections between different tubular neighbourhoods of the (many) different local unstable manifolds, and by the use of specification and the expression for equilibrium measure with the help of periodic points. This will imply also the use of some finer properties of the lifting of invariant measures to the natural extension and the comparison between the different types of behaviors of weighted sums of atomic measures on various prehistories, with respect to Gibbs states. The **main results** of the paper are in Theorem 1 and its Corollaries:

Theorem. *Let $f : M \rightarrow M$ be a smooth map (say \mathcal{C}^2) on a smooth Riemannian manifold M , so that f is hyperbolic and finite-to-one on a basic set Λ ; assume also that the critical set \mathcal{C}_f of f does not intersect Λ . Let also ϕ a Holder continous potential on Λ and μ_ϕ be the equilibrium measure of ϕ on Λ . Then*

$$\int_{\Lambda} | < \frac{1}{n} \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{e^{S_n\phi(y)}}{\sum_{z \in f^{-n}(x) \cap \Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \xrightarrow{n \rightarrow \infty} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R}).$$

In Corollary 1 we obtain an approximation for the equilibrium measure μ_ϕ , i.e the weak- \star convergence of a sequence of weighted atomic probabilities of the above type towards μ_ϕ .

Corollary. *In the same setting as in Theorem 1, for any Holder potential ϕ with equilibrium measure μ_ϕ , it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak- \star convergence of measures:*

$$\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{e^{S_{n_k}\phi(y)}}{\sum_{z \in f^{-n_k}(x) \cap \Lambda} e^{S_{n_k}\phi(z)}} \cdot \sum_{i=0}^{n_k-1} \delta_{f^i y} \xrightarrow{k \rightarrow \infty} \mu_\phi$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0 -almost all points $x \in \Lambda$,

$$\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{\sum_{i=0}^{n_k-1} \delta_{f^i y}}{\text{Card}(f^{-n_k}(x) \cap \Lambda)} \xrightarrow[k \rightarrow \infty]{} \mu_0, \text{ for a subsequence } (n_k)_k.$$

We remark that, since μ_ϕ is positive on any open set (as any open set contains some small Bowen ball and one can apply Proposition 1), there exists a dense set in Λ of points x for which we have the above weak convergence of weighted atomic measures generated by x , towards μ_ϕ . Therefore in a physical non-reversible system, if one knows the past trajectories of such a generic point x up to some high level n , then one can approximate the Gibbs state μ_ϕ as above.

One may also compare this result with the usual (forward) SRB measure in the case of diffeomorphisms (see for example [2], [15], [16]) or endomorphisms ([4], [12], [11]); in fact whenever μ_ϕ is equivalent to the Lebesgue measure, like in the case of toral endomorphisms and $\phi \equiv 0$, we obtain an *inverse SRB result*. Our setting and methods are however different due to the lack of an inverse function, the fact that unstable manifolds depend on whole prehistories (not just base points), and also to the fact that the number of preimages is not necessarily constant on Λ . We will apply the L^1 Ergodic Theorem of von Neumann on the natural extension $\hat{\Lambda}$ for the lifted measure, together with combinatorial arguments in order to estimate the measure μ_ϕ on the intersections between different tubular unstable sets. Then we will estimate the equilibrium measure on the different parts of the consecutive preimage sets in Λ , by carefully studying the prehistories from the point of view of convergence properties of certain weighted sums of Dirac measures along them. In our Theorem let us notice that we average over **all n -preimages** of points, so we do not consider only one prehistory. This **simultaneous** consideration of all n -preimages is what makes the proof difficult. It cannot be obtained just by applying Birkhoff Ergodic Theorem to different prehistories, since the speeds of convergence may be different over the *uncountable* collection of prehistories.

Among the **examples** of smooth endomorphisms on folded basic sets, let us mention the horse-shoes with self-intersections from [1], the hyperbolic skew products with overlaps in their fibers from [10], or dynamical systems generating from certain non-reversible statistical physics models (see [15]). Examples may be obtained also from non-degenerate holomorphic maps on complex projective spaces (for instance [9], [7]).

Another application of Theorem 1 will be in Corollary 2, where it will be applied to **Anosov endomorphisms**, in order to give the distribution of consecutive preimage sets, with respect to different equilibrium measures. A classical example of Anosov endomorphism is given by a toral endomorphism $f_A : \mathbb{T}^m \rightarrow \mathbb{T}^m$, $m \geq 2$, where f_A is the map induced on the m -dimensional torus by a matrix A with integer coefficients and $\det A \neq 0$. Then any point in \mathbb{T}^m has exactly $|\det A|$ f_A -preimages in \mathbb{T}^m , as can be seen since the f_A image of the unit square is a parallelogram with area (volume) equal to $|\det A|$, whose corners have integer coordinates ([17]). If A has all its eigenvalues of absolute values different from 1, then f_A is a hyperbolic endomorphism and the above Theorem will apply. Since the equilibrium measure of any constant function is the Haar measure ([17]), we obtain the asymptotic distribution of the local inverse iterates toward an *inverse SRB measure* in this case. A generalization of this class of examples is given by smooth **perturbations of**

hyperbolic toral endomorphisms on $\mathbb{T}^m, m \geq 2$. They will be again constant-to-1 and we will be able to apply our main Theorem, to obtain the weighted distribution of preimages with respect to equilibrium measures of Holder potentials.

Remark: Even on algebraic-type manifolds, like **infrnilmanifolds**, the situation of Anosov endomorphisms is very different from that of Anosov diffeomorphisms. Indeed in the case of infrnilmanifolds ([6], [18]), Franks and separately Manning showed that any Anosov diffeomorphism can be "linearized", i.e it is topologically conjugate to some hyperbolic automorphism. Also, Gromov ([6]) showed that if f is an expanding map on a compact manifold, then f is topologically conjugate to some expanding endomorphism on some infrnilmanifold. However this is not the case for Anosov endomorphisms. As was proved in [18], if M is an infrnilmanifold then there exists a \mathcal{C}^1 dense subset \mathcal{U} in the set of "true" Anosov endomorphisms on M (i.e those endomorphisms which are not Anosov diffeomorphisms nor expanding maps), such that every $f \in \mathcal{U}$ is not shift equivalent (hence also not topologically equivalent) to any hyperbolic infrnilmanifold endomorphism. In particular this applies to tori $\mathbb{T}^m, m \geq 2$, which are natural examples of infrnilmanifolds. For such Anosov endomorphisms which are neither diffeomorphisms nor expanding, one cannot apply results similar to the ones from those two previous cases. However we can apply Theorem 1 to get the distribution of preimage sets with respect to equilibrium measures.

2 Distributions of consecutive preimages on basic sets

First let us establish some notations. The next definition is parallel to that of basic set from [3] (see also [7]).

Definition 1. Let $f : M \rightarrow M$ a smooth map (say \mathcal{C}^2) defined on the smooth manifold M . We will say that a compact f -invariant set Λ is a **basic set** for f if there exists a neighbourhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$, and if f is transitive on Λ . As we work here with non-invertible maps, such sets will also be called **folded basic sets**.

Definition 2. The **natural extension** of the dynamical system (f, Λ) is the dynamical system $(\hat{f}, \hat{\Lambda})$, where $\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, x_{-2}, \dots), f(x_{-i}) = x_{-i+1}, x_0 = x, x_{-i} \in \Lambda, i \geq 1\}$ and $\hat{f}(\hat{x}) := (f(x), x, x_{-1}, \dots), \hat{x} \in \hat{\Lambda}$. It follows that \hat{f} is a homeomorphism on $\hat{\Lambda}$. An element $\hat{x} = (x, x_{-1}, \dots)$ of $\hat{\Lambda}$, starting with x , is called a **prehistory** of x (or *full prehistory* of x). The canonical projection $\pi : \hat{\Lambda} \rightarrow \Lambda$ is defined by $\pi(\hat{x}) = x, \hat{x} \in \hat{\Lambda}$. If $f(y) = x, y \in \Lambda$ we call y a **preimage** of x ; if $f^n(z) = x, z \in \Lambda$, we call z an **n -preimage** of x (through f). A finite sequence $(x, x_{-1}, \dots, x_{-n})$ will be called an **n -prehistory** of x .

Let us mention that if μ is an f -invariant probability measure on Λ , then there exists a unique probability \hat{f} -invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}) = \mu$. It can be seen that μ is ergodic if and only if $\hat{\mu}$ is ergodic on $\hat{\Lambda}$. Also the topological pressure of ϕ (denoted by $P_f(\phi)$ to emphasize dependence on f) is equal to the topological pressure of $\phi \circ \pi$, namely $P_{\hat{f}}(\phi \circ \pi)$; and μ is an equilibrium measure for $\phi : \Lambda \rightarrow \mathbb{R}$ if and only if $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$.

The concept of hyperbolicity on Λ makes sense for non-invertible maps (i.e endomorphisms), but now the unstable tangent subspaces and the local unstable manifolds depend on whole prehistories, not only on the base points (for exp. [14]). In this hyperbolic setting we will denote by $E_x^s, W_r^s(x)$ the stable tangent subspace, respectively the local stable manifold at x (for $x \in \Lambda$); and by $E_{\hat{x}}^u, W_r^u(\hat{x})$ the unstable tangent subspace, respectively the local unstable manifold corresponding to the pre-history \hat{x} (for $\hat{x} \in \hat{\Lambda}$). Also by $Df_s(x)$ we shall denote the stable derivative $Df_x|_{E_x^s}$ and by $Df_u(\hat{x})$ the unstable derivative $Df_x|_{E_{\hat{x}}^u}$. In [8] we studied the Holder dependence of local unstable manifolds with respect to the prehistories and proved a Bowen type formula, giving the unstable dimension as being the zero t^u of the pressure of the unstable potential $\Phi^u(\hat{x}) := -\log |Df_u(\hat{x})|$, $\hat{x} \in \hat{\Lambda}$. We proved also that given a measurable partition of $\hat{\Lambda}$ subordinated to the unstable manifolds, the equilibrium measure of Φ^u has conditional measures which are geometric of exponent t^u .

If $f : M \rightarrow M$ is a smooth map which is hyperbolic on a basic set Λ , let $\hat{x} = (x, x_{-1}, \dots, x_{-n}, \dots) \in \hat{\Lambda}$, $n \geq 1, \varepsilon > 0$ small. Then we call an (n, ε) -**tubular unstable neighbourhood** (or tubular unstable set) the set

$$T_n(\hat{x}, \varepsilon) := \{y \in \Lambda, \exists \text{ an } n\text{-preimage } y_{-n} \in \Lambda \text{ of } y, \text{ s.t } d(f^i y_{-n}, f^i x_{-n}) < \varepsilon, i = 0, \dots, n\}$$

The notion of tubular unstable set can be extended to those $y \in M$ which have an n -preimage $y_{-n} \in M$ with the above property, but since we will work in this paper only with measures supported on Λ , we preferred to give the definition restricted to Λ . It is important to keep in mind that tubular unstable sets corresponding to two different prehistories of the same point $x \in \Lambda$ *may not be the same*; still they intersect in a set containing x .

It is well-known that any f -invariant measure μ on Λ can be lifted to a unique \hat{f} -invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}) = \mu$. It will be important to see exactly how to calculate the measure $\hat{\mu}$ of an arbitrary closed set from $\hat{\Lambda}$, in terms of the μ -measures of sets in Λ .

Lemma 1. *Let $f : \Lambda \rightarrow \Lambda$ be a continuous map on a compact metric space Λ , and μ an f -invariant borelian probability measure on Λ . Let $\hat{\mu}$ be the unique \hat{f} -invariant probability measure on $\hat{\Lambda}$ with the property that $\pi_*(\hat{\mu}) = \mu$. Then for an arbitrary closed set $\hat{E} \subset \hat{\Lambda}$, we have that*

$$\hat{\mu}(\hat{E}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} = (x, \dots, x_{-n}, \dots) \in \hat{E}\})$$

Proof. The arbitrary closed set \hat{E} is not necessarily of the form $\pi^{-1}(E)$ for some E borelian set in Λ . Let us denote $\hat{E}_n := \hat{f}^{-n}\hat{E}, n \geq 1$; then $\hat{\mu}(\hat{E}_n) = \hat{\mu}(\hat{E})$ since $\hat{\mu}$ is \hat{f} -invariant. Let also $\hat{F}_n := \pi^{-1}(\pi(\hat{E}_n)), n \geq 1$. We will prove that

$$\hat{E} = \bigcap_{n \geq 0} \hat{f}^n(\hat{F}_n)$$

We have clearly $\hat{E} \subset \hat{f}^n(\hat{F}_n), n \geq 0$. Let now a prehistory $\hat{z} \in \bigcap_{n \geq 0} \hat{f}^n \hat{F}_n$; then if $\hat{z} = (z, z_{-1}, \dots, z_{-n}, \dots)$, we obtain that $z_{-n} \in \pi \hat{E}_n, \forall n \geq 0$, hence $\hat{z} \in \hat{E}$ since \hat{E} is assumed closed. Thus we showed the equality $\hat{E} = \bigcap_{n \geq 0} \hat{f}^n(\hat{F}_n)$. Now let us notice that the above intersection is decreasing, since $\hat{f}^{n+1} \hat{F}_{n+1} \subset \hat{f}^n \hat{F}_n, n \geq 0$, since for a prehistory from $\hat{f}^{n+1} \hat{F}_{n+1}$ the $(n+1)$ -th entry is from $\pi \hat{E}_{n+1}$

and the n -th entry is in $\pi\hat{E}_n$, whereas the $(n+1)$ -entry of a prehistory from $\hat{f}^n\hat{F}_n$ can be **any** preimage of a point from $\pi\hat{E}_n$. Since the above intersection is decreasing, we get

$$\hat{\mu}(\hat{E}) = \lim_n \hat{\mu}(\hat{f}^n\hat{F}_n) = \lim_n \hat{\mu}(\hat{F}_n) = \lim_n \hat{\mu}(\pi^{-1}(\pi(\hat{E}_n))) = \lim_n \mu(\pi(\hat{E}_n)) = \lim_n \mu(\pi \circ \hat{f}^{-n}\hat{E})$$

We used that $\pi_*\hat{\mu} = \mu$ and that $\hat{\mu}$ is \hat{f} -invariant on $\hat{\Lambda}$. Therefore we obtain that $\hat{\mu}(\hat{E}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} \in \hat{E}, \hat{x} = (x, \dots, x_{-n}, \dots)\})$.

□

For a basic set Λ for a smooth map f we will denote by $f_\Lambda^{-1}x, x \in \Lambda$ the set of f -preimages of x **which belong to** Λ . Similarly $f_\Lambda^{-n}x$ will denote the n -preimages of x belonging to Λ , i.e $f_\Lambda^{-n}x := f^{-n}x \cap \Lambda$. In general in this paper we will be interested only in the preimages belonging to Λ . We will denote also by $S_n\phi(y)$ for a point $y \in \Lambda$, the **consecutive sum** $S_n\phi(y) := \phi(y) + \dots + \phi(f^{n-1}y), n \geq 1$. Define then for $n \geq 1$ and $x \in \Lambda$, the probability measure:

$$\mu_n^x := \frac{1}{n} \sum_{y \in f_\Lambda^{-n}x} \frac{e^{S_n\phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} \quad (1)$$

The $\mu_n^x, n \geq 1$ are probability measures and thus from the weak compactness of the unit ball in the space of measures, we obtain that any sequence of such measures (for $x \in \Lambda$ given) contains a convergent subsequence; the limit of such a sequence $(\mu_{n_k}^x)_{k \geq 1}$ is then an f -invariant probability measure on Λ . We will show that such limit measures have in fact an important thermodynamical property, namely they are equilibrium measures (i.e maximize in the Variational Principle, [17]).

We shall use in the sequel **equilibrium measures for endomorphisms**; the existence of these measures for non-invertible maps, and estimates on Bowen balls for these measures are similar to the corresponding properties for diffeomorphisms (see the proof of Proposition 1 below). We denote the Bowen ball $B_n(y, \varepsilon) := \{z \in \Lambda, d(f^i y, f^i z) < \varepsilon, i = 0, \dots, n-1\}$, for $y \in \Lambda, n \geq 1, \varepsilon > 0$.

Proposition 1. *Let Λ be a hyperbolic basic set for a smooth endomorphism $f : M \rightarrow M$, and ϕ be a Holder continuous function on Λ . Then there exists a unique equilibrium measure μ_ϕ for ϕ on Λ which has the following properties:*

a) *for any $\varepsilon > 0$ there exist positive constants $A_\varepsilon, B_\varepsilon$ and an integer $n_0 \geq 1$ such that for any $y \in \Lambda, n \geq n_0$,*

$$A_\varepsilon e^{S_n\phi(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n\phi(y) - nP(\phi)}$$

b) *By working eventually with a finite iteration of f , we have*

$$\mu_\phi = \lim_{n \rightarrow \infty} \frac{1}{P_\Lambda(f, \phi, n)} \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n\phi(x)} \delta_x,$$

where $P_\Lambda(f, \phi, n) := \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n\phi(x)}, n \geq 1$.

Proof. a) We work in the natural extension $\hat{\Lambda}$ with the expansive homeomorphism $\hat{f} : \hat{\Lambda} \rightarrow \hat{\Lambda}$. The existence of a unique equilibrium measure for the Holder potential $\phi \circ \pi$ with respect to the expansive homeomorphism $\hat{f} : \hat{\Lambda} \rightarrow \hat{\Lambda}$ follows from the standard theory for homeomorphisms on compact metric spaces (see for example [2], [3]); let us denote this equilibrium measure by $\hat{\mu}_\phi$. Then, given an \hat{f} -invariant probability measure $\hat{\mu}$ on $\hat{\Lambda}$, there exists a unique f -invariant measure μ on Λ such that $\pi_*\hat{\mu} = \mu$. If we take the measure $\hat{\mu}_\phi$ instead of $\hat{\mu}$, we will obtain a measure μ_ϕ . It is easy to show that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$ and that $P_{\hat{f}}(\phi \circ \pi) = P_f(\phi), \forall \phi \in \mathcal{C}(\Lambda, \mathbb{R})$. Thus it follows that μ is an equilibrium measure for ϕ if and only if its unique \hat{f} -invariant lifting $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$ on $\hat{\Lambda}$. Thus $\mu_\phi := \pi_*\hat{\mu}_\phi$ is the unique equilibrium measure for ϕ on Λ . Now we see that there exists a $k = k(\varepsilon) \geq 1$ such that $\hat{f}^k(\pi^{-1}B_n(y, \varepsilon)) \subset B_{n-k}(\hat{f}^k\hat{y}, 2\varepsilon) \subset \hat{\Lambda}$, for any $y \in \Lambda$. On the other hand for any $\hat{y} \in \hat{\Lambda}$, we have $\pi(B_n(\hat{y}, \varepsilon)) \subset B_n(y, \varepsilon)$. These two inclusions and the \hat{f} -invariance of $\hat{\mu}_\phi$, together with the estimates for the $\hat{\mu}_\phi$ -measure of the Bowen balls in $\hat{\Lambda}$ (from [3]) imply that there exist positive constants $A_\varepsilon, B_\varepsilon$ (depending on $\varepsilon > 0$ and on ϕ) such that:

$$A_\varepsilon e^{S_n(\phi)(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n(\phi)(y) - nP(\phi)}, \forall y \in \Lambda, n \geq 1$$

So the estimates for Bowen balls are true also for endomorphisms.

b) The iterate of f may be needed in order to have topological mixing (needed to guarantee specification, [3]). However without loss of generality we may assume that f is topologically mixing on Λ . If x is a periodic point for $f|_\Lambda$, say $f^m(x) = x$, then we obtain a periodic point for \hat{f} , namely the prehistory $\hat{x} = (x, f^{m-1}(x), \dots, f(x), x, \dots, f(x), x, \dots) \in \hat{\Lambda}$. Conversely, if \hat{x} is a periodic point for \hat{f} , then x is a periodic point (of the same period) for f . Similarly as for diffeomorphisms we prove that if f is hyperbolic then f satisfies specification. Then specification is used to show the convergence of the weighted sums of Dirac measures concentrated at periodic points towards μ_ϕ , in the same way as in [3].

□

Let us prove now a Lemma giving a relationship between the measures of different parts of the preimage of some set of positive measure.

Lemma 2. *In the above setting, let a Holder potential $\phi : \Lambda \rightarrow \mathbb{R}$ and its unique equilibrium measure μ_ϕ . Let us consider $\varepsilon > 0$, k disjoint Bowen balls $B_m(y_1, \varepsilon), \dots, B_m(y_k, \varepsilon)$ and a borelian set $A \subset f^m B_m(y_1, \varepsilon) \cap \dots \cap f^m B_m(y_k, \varepsilon)$ such that $\mu_\phi(A) > 0$; denote by $A_1 := f^{-m}A \cap B_m(y_1, \varepsilon), \dots, A_k := f^{-m}A \cap B_m(y_k, \varepsilon)$. Then there exists a positive constant C_ε independent of m, y_1, \dots, y_m such that*

$$\frac{1}{C_\varepsilon} \mu_\phi(A_j) \cdot \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}} \leq \mu_\phi(A_i) \leq C_\varepsilon \mu_\phi(A_j) \cdot \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}}, i, j = 1, \dots, m$$

Proof. We shall denote the equilibrium measure μ_ϕ by μ to simplify notation, and will work with the restriction of f to Λ (without saying this explicitly every time).

Similarly as in [3] or in [17], since the borelian sets with boundaries of measure zero form a sufficient collection, we may assume that each of the sets A_i, A_j have boundaries of μ -measure zero.

From construction $f^m(A_i) = f^m(A_j)$, $i, j = 1, \dots, k$. But μ can be considered as the limit of the sequence of measures

$$\tilde{\mu}_n := \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)} \delta_x,$$

where $P(f, \phi, n) := \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)}$, $n \geq 1$. So we have

$$\tilde{\mu}_n(A_i) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap A_i} e^{S_n \phi(x)}, n \geq 1 \quad (2)$$

Let us consider now a periodic point $x \in \text{Fix}(f^n) \cap A_i$; by definition of A_i , it follows that $f^m(x) \in A_i$, so there exists a point $y \in A_j$ such that $f^m(y) = f^m(x)$. Of course the point y does not have to be periodic necessarily. But we can use Specification Property ([3], [2]) on the hyperbolic compact locally maximal set Λ . Indeed if $\varepsilon > 0$ is fixed, then there exists a constant $M_\varepsilon > 0$ such that for all $n > M_\varepsilon$, there is a $z \in \text{Fix}(f^n)$ s.t z ε -shadows the $(n - M_\varepsilon)$ -orbit of y . Thus $z \in B_m(y_i, 2\varepsilon)$ if y_i is so that $y \in B_m(y_i, \varepsilon)$.

Let now V be an arbitrary neighbourhood of the set A_j . Then if n is large enough and $\varepsilon > 0$ fixed, it follows that $z \in V$. Let us also take two points $x, \tilde{x} \in \text{Fix}(f^n) \cap A_i$ and assume the same periodic point $z \in V \cap \text{Fix}(f^n)$ corresponds to both through the previous procedure. This means that the $(n - M_\varepsilon - m)$ -orbit of $f^m z$ ε -shadows the $(n - M_\varepsilon - m)$ -orbit of $f^m x$ and also the $(n - M_\varepsilon - m)$ -orbit of $f^m \tilde{x}$. Hence the $(n - M_\varepsilon - m)$ -orbit of $f^m x$ 2ε -shadows the $(n - M_\varepsilon - m)$ -orbit of $f^m \tilde{x}$. But recall that we chose $x, \tilde{x} \in A_i \subset B_m(y_i, \varepsilon)$, hence $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$.

Now we can split the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ in at most N_ε smaller Bowen ball of type $B_n(\zeta, 2\varepsilon)$. In each of these $(n, 2\varepsilon)$ -Bowen balls we may have at most one fixed point for f^n . This holds since fixed points for f^n are solutions to the equation $f^n \xi = \xi$ and, on tangent spaces we have that $Df^n - Id$ is a linear map without eigenvalues of absolute value 1. Thus if $d(f^i \xi, f^i \zeta) < 2\varepsilon$, $i = 0, \dots, n$ and if ε is small enough, it follows that we can apply the Inverse Function Theorem at each step. Therefore there will exist only one fixed point for f^n in the Bowen ball $B_n(\zeta, 2\varepsilon)$. So there may exist at most N_ε periodic points from $\text{Fix}(f^n) \cap \Lambda$ which have the same point $z \in V$ attached to them by the above procedure. In conclusion, to each point $x \in \text{Fix}(f^n) \cap A_i$, there corresponds at most finitely many points $z \in V$ obtained by specification from the above procedure; their number is smaller than N_ε . Let us notice also that if x, \tilde{x} have the same point $z \in V$ attached to them, then as seen before $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$ and thus, from the Holder continuity of ϕ :

$$|S_n \phi(x) - S_n \phi(\tilde{x})| \leq \tilde{C}_\varepsilon,$$

for some positive constant \tilde{C}_ε depending on ϕ (but independent of n, x). This can be used then in the estimate for $\tilde{\mu}_n(A_i)$, according to (2). Now we use the fact that if $z \in B_{n-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-m}(f^m x, \varepsilon)$. From the Holder continuity of ϕ and the fact that $x \in A_i \subset B_m(y_i, \varepsilon)$, it follows that there exists a constant \tilde{C}_ε (we denote it the same as before, without loss of generality) satisfying:

$$|S_n \phi(z) - S_n \phi(x)| \leq |S_m \phi(y_i) - S_m \phi(y_j)| + \tilde{C}_\varepsilon, \quad (3)$$

for $n > n(\varepsilon, m)$. This, together with (2) and the fact that there are at most N_ε points $x \in \text{Fix}(f^n)$ which have the same attached $z \in V \cap \text{Fix}(f^n)$, imply that there exists a constant $C_\varepsilon > 0$ such that

$$\tilde{\mu}_n(A_i) \leq C_\varepsilon \tilde{\mu}_n(V) \cdot \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}}, \quad (4)$$

where we recall that $A_i \subset B_m(y_i, \varepsilon)$, $A_j \subset B_m(y_j, \varepsilon)$. But since $\partial A_i, \partial A_j$ have μ -measure zero, we obtain:

$$\mu(A_i) \leq C_\varepsilon \mu(V) \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}}, i, j = 1, \dots, k$$

Now V was chosen arbitrarily as a neighbourhood of A_j , thus

$$\mu(A_i) \leq C_\varepsilon \mu(A_j) \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}}, i, j = 1, \dots, k$$

Similarly we prove also the other inequality. □

Theorem 1. *Let $f : M \rightarrow M$ be a smooth (say \mathcal{C}^2) map on a Riemannian manifold M , which is hyperbolic and finite-to-one on a basic set Λ so that $C_f \cap \Lambda = \emptyset$. Assume that ϕ is a Holder continuous potential on Λ and that μ_ϕ is the equilibrium measure of ϕ on Λ . Then with the notation from (1),*

$$\int_\Lambda | \langle \mu_n^x - \mu_\phi, g \rangle | d\mu_\phi(x) \xrightarrow{n \rightarrow \infty} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

Proof. We make the convention that all the preimages that we work with are in Λ . So we shall write $f^{-n}x$ for $f_\Lambda^{-n}x$, $n \geq 1, x \in \Lambda$.

If ϕ is a Holder continuous function on Λ it follows from Proposition 1 that there exists a unique equilibrium measure μ_ϕ for ϕ , and μ_ϕ is the push-forward of the equilibrium measure $\hat{\mu}_\phi$ of $\phi \circ \pi$ on $\hat{\Lambda}$. For simplicity of notation, we shall denote the measure μ_ϕ by μ , with ϕ being fixed. This measure is ergodic as being an equilibrium measure.

Let us fix now a continuous test function $g : \Lambda \rightarrow \mathbb{R}$. From von Neumann's L^1 Ergodic Theorem applied to the homeomorphism $\hat{f}^{-1} : \hat{\Lambda} \rightarrow \hat{\Lambda}$ and the potential $g \circ \pi$, we know that

$$\int_{\hat{\Lambda}} \left| \frac{1}{n} \sum_{i=0}^{n-1} g(x_{-i}) - \int_{\hat{\Lambda}} g \circ \pi d\hat{\mu} \right| d\hat{\mu}(\hat{x}) \xrightarrow{n \rightarrow \infty} 0, \quad (5)$$

where the prehistory $\hat{x} = (x, x_{-1}, x_{-2}, \dots, x_{-i}, \dots) \in \hat{\Lambda}$. Denote by

$$\Sigma_n(g, y) := \frac{\sum_{i=0}^{n-1} g(f^i y)}{n} - \int_\Lambda g d\mu, y \in \Lambda, n \geq 2$$

Hence for an arbitrary small $\eta > 0$, we have from (5) that:

$$\hat{\mu}(\{\hat{x} = (x, x_{-1}, \dots) \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\}) \xrightarrow{n \rightarrow \infty} 0$$

So for any $\varepsilon' > 0, \varepsilon' = \varepsilon'(\eta) \ll \eta$, there exists $n(\eta) \geq 1$ so that if $n > n(\eta)$ then

$$\hat{\mu}(\hat{x}, |\Sigma_n(g, x_{-n})| \geq \eta) < \varepsilon'$$

But now $\{\hat{x} \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\}$ is a closed set in $\hat{\Lambda}$, thus we can apply Lemma 1 to prove that:

$$\mu(x_{-n} \in \Lambda, |\Sigma_n(g, x_{-n})| \geq \eta) < \varepsilon', \quad (6)$$

if n is large enough (without loss of generality we can take $n > n(\eta)$).

Now let us consider a small $\varepsilon > 0$ with $\varepsilon < \varepsilon(\eta) \ll \eta$ such that $\omega(3\varepsilon) < \eta$, where $\omega(r)$ denotes in general the maximal oscillation of g on a ball of radius $r > 0$. Let us take also a maximal set of mutually disjoint n -Bowen balls $B_n(y, \varepsilon)$ in Λ ; denote the set of such y by \mathcal{F}_n . Thus $\{B_n(y, \varepsilon), y \in \mathcal{F}_n\}$ is our maximal set. If $z \notin \mathcal{F}_n$, then from the definition, $B_n(z, \varepsilon)$ must intersect some Bowen ball $B_n(y, \varepsilon), y \in \mathcal{F}_n$. Thus $B_n(z, \varepsilon) \subset B_n(y, 3\varepsilon)$. Let us notice also that if $w \in B_n(z, 3\varepsilon)$ then

$$|\Sigma_n(g, w)| \leq |\Sigma_n(g, z)| + \omega(3\varepsilon) \quad (7)$$

In the sequel we will split different subsets of \mathcal{F}_n in two disjoint subsets $\mathcal{R}_n, \mathcal{G}_n$, with $\mathcal{R}_n \subset \{x \in \Lambda, |\Sigma_n(g, x)| \geq 2\eta\}$ and $\mathcal{G}_n \subset \{x \in \Lambda, |\Sigma_n(g, x)| < 2\eta\}$. Intuitively \mathcal{R}_n consists of the "bad" n -preimages (corresponding to g, η) and \mathcal{G}_n are the "good" n -preimages.

Recall now that we denoted $f_\Lambda^{-n}x := \Lambda \cap f^{-n}x, n \geq 1, x \in \Lambda$, and that Λ is not necessarily totally invariant. Consider then

$$I_n(g, x) := \sum_{y \in f_\Lambda^{-n}x} \frac{e^{S_n \phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n \phi(z)}} \cdot |\Sigma_n(g, y)|, x \in \Lambda, n \geq 1$$

Denote also by

$$\Lambda(n, \eta) := \{x \in \Lambda, \text{ s. t } x \text{ has at least one } n - \text{preimage } x_{-n} \text{ with } |\Sigma_n(g, x_{-n})| \geq 2\eta\}$$

The problem is that the unstable manifolds may depend on the whole prehistories, thus if we take $z, w \in f^{-n}x$, then $f^n B_n(w, \varepsilon)$ and $f^n B_n(z, \varepsilon)$ may be different unstable tubular sets; these unstable tubular sets intersect each other in a (possibly smaller) set containing x . By taking n -preimages for all points in \mathcal{F}_n and then the corresponding tubular unstable sets as above, we shall obtain a collection of such tubular unstable sets which intersect each other in different smaller pieces, denoted generically by D . We have to estimate $\int_\Lambda I_n(g, x) d\mu(x)$, which is a sum of integrals on sets of type D . The problematic terms in this sum are those of type $\frac{e^{S_n \phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n \phi(z)}} \cdot |\Sigma_n(g, y)| \mu(D)$,

with $|\Sigma_n(g, y)| \geq 2\eta$, but so that **at the same time** there exist also good n -preimages for every point $x \in D$. In other words $x \in \Lambda(n, \eta)$, but x has also good n -preimages. The set of points where **all** n -preimages from Λ are bad, is easier to measure, by using Lemma 1. In the integral $\int_\Lambda I_n(g, x) d\mu(x)$ we have to deal both with the measures of subsets D , and with the n -preimages y of points $x \in D$, namely with the quantities $|\Sigma_n(g, y)|$.

For the measures of subsets D we shall use the f -invariance of μ , namely $\mu(D) = \mu(f^{-n}(D))$. Then we will use the estimate on the measure of the set of bad preimages from Lemma 1, coupled with the existent control on the good preimages given by $|\Sigma_n(g, y)| < \eta$. For the rest of the preimages, the idea is to control the sum of the measures of these inverse iterates by the measure of the set of bad preimages.

First let us notice that if $x \in \Lambda$ and $y, z \in f^{-1}x \cap \Lambda, y \neq z$, then since the critical set \mathcal{C}_f does not intersect Λ , it must exist a positive constant ε_0 so that $d(y, z) > \varepsilon_0$. Hence if similarly for some $n \geq 1$, we take two distinct n -preimages $y, z \in \Lambda$ of x ($f^n y = f^n z = x, y \neq z$), then we cannot have $y, z \in B_n(w, 3\varepsilon)$ for any $w \in \Lambda$, if $\varepsilon < \varepsilon_0$. We know also that the Bowen balls $B_n(y, 3\varepsilon)$ cover Λ when y ranges in \mathcal{F}_n , and $B_n(y_i, \varepsilon) \cap B_n(y_j, \varepsilon) = \emptyset$ for any two different points y_i, y_j from \mathcal{F}_n .

Let us take then a small $0 < \beta < 1$ such that $\eta < \beta$; to fix ideas we will consider $\beta = 3\eta$. If $x \in \Lambda$, denote by $R_n(x)$ the set of n -preimages $y \in \Lambda$ of x with $|\Sigma_n(g, y)| > 2\eta$ (in fact $R_n(x)$ depends on the η as well, but we do not record this here in order to simplify notation). Let us denote now

$$D_n(\beta, \eta) := \{x \in \Lambda(n, \eta), \frac{\sum_{y \in R_n(x)} e^{S_n \phi(y)}}{\sum_{y \in f^{-n}x} e^{S_n \phi(y)}} < \beta\} \quad (8)$$

Now if $\|g\| := \sup_{y \in \Lambda} |g(y)|$, it follows from definition that for any $y \in \Lambda$, we have $|\Sigma_n(g, y)| \leq 2\|g\|$; thus for a point $x \in D_n(\beta, \eta)$:

$$\begin{aligned} \sum_{y \in f_\Lambda^{-n}x} \frac{e^{S_n \phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n \phi(z)}} |\Sigma_n(g, y)| &= \sum_{y \in R_n(x)} \frac{e^{S_n \phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n \phi(z)}} |\Sigma_n(g, y)| + \sum_{y \in f_\Lambda^{-n}x \setminus R_n(x)} \frac{e^{S_n \phi(y)}}{\sum_{z \in f_\Lambda^{-n}x} e^{S_n \phi(z)}} |\Sigma_n(g, y)| \\ &\leq 2\|g\| \cdot \beta + \eta \end{aligned}$$

Therefore

$$\int_{D_n(\beta, \eta)} I_n(g, x) d\mu(x) \leq 2\|g\|\beta + 2\eta \quad (9)$$

We will now restrict to the complement of $D_n(\beta, \eta)$ in Λ , so we work with points $x \in \Lambda$ for which $\frac{\sum_{y \in R_n(x)} e^{S_n \phi(y)}}{\sum_{y \in f^{-n}x} e^{S_n \phi(y)}} \geq \beta$. From Proposition 1 we know that for any $z \in \Lambda, n \geq 1$, $A_\varepsilon \cdot e^{S_n \phi(z) - nP(\phi)} \leq \mu(B_n(z, \varepsilon)) \leq B_\varepsilon \cdot e^{S_n \phi(z) - nP(\phi)}$. Therefore we have

$$\sum_{y \in f_\Lambda^{-n}x} \mu(B_n(y, \varepsilon)) \leq \frac{C_\varepsilon}{\beta} \cdot \sum_{z \in R_n(x)} \mu(B_n(z, \varepsilon)) \quad (10)$$

Obviously from the definition of \mathcal{F}_n we can place any n -preimage of x in a distinct Bowen ball of type $B_n(z, 3\varepsilon)$, for some $z \in \mathcal{F}_n$. From Proposition 1 it follows that the μ -measure of a Bowen ball of type $B_n(z, \varepsilon)$ is comparable to the μ -measure of the Bowen ball $B_n(z, 3\varepsilon)$ (by **comparable** we mean that their quotient is bounded below and above by a positive constant independent of

$z, n)$. The idea is to split now the set $\Lambda \setminus D_n(\beta, \eta)$ into subsets A such that points from A have the same number of n -preimages. Let us assume that d is the maximum number of 1-preimages that a point may have in Λ . Then any point from Λ has at most d^n n -preimages in Λ . We will use now tubular unstable sets $T_n(\hat{x}, 3\varepsilon)$ obtained from different Bowen balls centered at the points of \mathcal{F}_n . For an integer $\ell \geq 2$ let us consider the sets of type

$$f^n(B_{i_1}) \cap \dots \cap f^n(B_{i_\ell}) \setminus \bigcup_{\ell < |J| \leq d^n, J \subset \mathcal{F}_n} \bigcap_{j \in J} f^n(B_j),$$

where $B_j := B_n(y_j, 3\varepsilon)$ for some points $y_j \in \mathcal{F}_n$, and where we assume no repetitions among the respective Bowen balls. This means actually that we do not repeat a Bowen ball B_j in the above intersection; if y, z are different n -preimages of the same point x and ε is small enough, then we cannot have y, z in the same $B_n(\zeta, 3\varepsilon)$. Each point in such a set has exactly ℓ n -preimages, one in each of the Bowen balls $B_{i_1}, \dots, B_{i_\ell}$.

Let us denote the collection of all such sets by $F(\ell, n, \varepsilon)$ and let $D \in F(\ell, n, \varepsilon)$. Consider the sets of type $D \setminus \bigcup_{D' \in F(\ell, n, \varepsilon), D' \neq D} D'$, $D \in F(\ell, n, \varepsilon)$. These sets are now mutually disjoint, borelian, and cover Λ . Their collection will be denoted by $\tilde{F}(\ell, n, \varepsilon)$.

We want to estimate the measure $\mu(\Lambda \setminus D_n(\beta, \eta))$. In order to do this, we will split $\Lambda \setminus D_n(\beta, \eta)$ into mutually disjoint borelian subsets, obtained by intersecting the sets of $\tilde{F}(\ell, n, \varepsilon)$ with $\Lambda \setminus D_n(\beta, \eta)$. Let us denote the collection of these intersections by $H(\ell, n, \varepsilon)$, $\ell \geq 1$.

We will take an arbitrary subset $S \in H(\ell, n, \varepsilon)$, say $S \subset f^n(B_{i_1}) \cap \dots \cap f^n(B_{i_\ell})$. From the f -invariance of μ , it follows that $\mu(S) = \mu(f^{-n}S) = \mu(f^{-n}S \cap B_{i_1}) + \dots + \mu(f^{-n}S \cap B_{i_\ell})$. We have that the Bowen balls B_{i_s} , $s = 1, \dots, \ell$ are mutually disjoint as they contain different n -preimages of the same point (we know that for small ε one cannot have two different n -preimages of the same point, belonging to the same $B_n(y, 3\varepsilon)$, $y \in \Lambda$). Let us denote by $S_n(i_1) := f^{-n}S \cap B_{i_1}, \dots, S_n(i_k) := f^{-n}S \cap B_{i_k}$. So

$$\mu(S) = \mu(S_n(i_1)) + \dots + \mu(S_n(i_k))$$

We assume that $\mu(S_n(i_1)) > 0$, otherwise we can take a different $S_n(j)$. But now from Lemma 2 and the fact that $S \subset \Lambda \setminus D_n(\beta, \eta)$, it follows that

$$\begin{aligned} \mu(S_n(i_1)) + \dots + \mu(S_n(i_k)) &\leq C_\varepsilon \frac{\mu(S_n(i_1))}{\mu(B_{i_1})} [\mu(B_{i_1}) + \dots + \mu(B_{i_k})] \leq \\ &\leq C_\varepsilon \frac{\mu(S_n(i_1))}{\mu(B_{i_1})} \cdot \frac{1}{\beta} \sum_{B_j \in R_n(x), j \in \{i_1, \dots, i_k\}} \mu(B_j) \end{aligned} \tag{11}$$

Suppose that T is another disjoint set from some $H(p, n, \varepsilon)$, so that $T \subset f^n(B_{i_1}) \cap f^n(B_{j_2}) \cap \dots \cap f^n(B_{j_p})$ and let $T_n(i_1), T_n(j_2), \dots, T_n(j_p)$ be the corresponding parts of $f^{-n}T$ belonging respectively to $B_{i_1}, B_{j_2}, \dots, B_{j_p}$. So the sets $S_n(i_1), \dots, S_n(i_\ell), T_n(i_1), T_n(j_2), \dots, T_n(j_p)$ are mutually disjoint borelian subsets. We assumed that S and T have both n -preimages in B_{i_1} (for example). If they have n -preimages in completely different Bowen balls, then the situation will be simpler, since there will be no repetitions below in (12).

Let us estimate now $\mu(S) + \mu(T)$; for this we consider two points $x \in S$ and $y \in T$ and assume that $\{i_1, \dots, i_\ell\} \cap \{i_1, j_2, \dots, j_p\} \cap R_n(x) = \{l_1, \dots, l_r\}$. Then from (11), we obtain:

$$\begin{aligned} \mu(S) + \mu(T) &\leq \frac{C_\varepsilon}{\beta} \left[\frac{\mu(S_n(i_1))}{\mu(B_{i_1})} \cdot \sum_{B_j \in R_n(x), j \in \{i_1, \dots, i_\ell\}} \mu(B_j) + \frac{\mu(T_n(i_1))}{\mu(B_{i_1})} \cdot \sum_{B_s \in R_n(y), s \in \{i_1, j_2, \dots, j_p\}} \mu(B_s) \right] \\ &\leq \frac{C_\varepsilon}{\beta \mu(B_{i_1})} \cdot [\mu(S_n(i_1)) + \mu(T_n(i_1))] \cdot [\mu(B_{l_1}) + \dots + \mu(B_{l_r})] + \\ &\quad + C_\varepsilon \cdot \frac{\mu(S_n(i_1))}{\beta \mu(B_{i_1})} \cdot \Sigma(S, n) + C_\varepsilon \cdot \frac{\mu(T_n(i_1))}{\beta \mu(B_{i_1})} \cdot \Sigma(T, n), \end{aligned} \tag{12}$$

where

$$\Sigma(S, n) := \sum_{j \in R_n(x) \cap \{i_1, \dots, i_\ell\} \setminus \{l_1, \dots, l_r\}} \mu(B_j) \quad \text{and} \quad \Sigma(T, n) := \sum_{j' \in \{i_1, j_2, \dots, j_p\} \cap R_n(x) \setminus \{l_1, \dots, l_r\}} \mu(B_{j'})$$

But recall that the subsets $S_n(i_1)$ and $T_n(i_1)$ are disjoint inside B_{i_1} , hence $\mu(S_n(i_1)) + \mu(T_n(i_1)) \leq \mu(B_{i_1})$. Since $\{l_1, \dots, l_r\} = \{i_1, \dots, i_\ell\} \cap \{i_1, j_2, \dots, j_p\} \cap R_n(x)$, the sums $\Sigma(S, n)$ and $\Sigma(T, n)$ do not have common terms and do not have any term from the collection $\{B_{l_1}, \dots, B_{l_r}\}$. Thus from (12) we obtain that

$$\mu(S) + \mu(T) \leq \frac{C_\varepsilon}{\beta} [\mu(B_{l_1}) + \dots + \mu(B_{l_r}) + \Sigma(S, n) + \Sigma(T, n)]$$

From (7) we obtain that $B_i \subset \{y \in \Lambda, |\Sigma_n(g, y)| \geq \eta\}$. Also from the estimates of Proposition 1, we know that there exists a positive constant χ_ε such that $\mu(B_i) \leq \chi_\varepsilon \mu(B_n(y_i, \varepsilon))$, $i \geq 1$. Therefore, recalling also that the balls $B_n(y_i, \varepsilon)$, $y_i \in \mathcal{F}_n$ are mutually disjoint we have that

$$\begin{aligned} \sum_{1 \leq k \leq r} \mu(B_{l_k}) + \Sigma(S, n) + \Sigma(T, n) &\leq \chi_\varepsilon \cdot \left[\sum_{1 \leq k \leq r} \mu(B_n(y_{l_k}, \varepsilon)) + \sum_{j \in R_n(x) \cap \{i_1, \dots, i_\ell\} \setminus \{l_1, \dots, l_r\}} \mu(B_n(y_j, \varepsilon)) \right] + \\ &+ \sum_{j' \in R_n(x) \cap \{i_1, j_2, \dots, j_p\} \setminus \{l_1, \dots, l_r\}} \mu(B_n(y_{j'}, \varepsilon)) = \chi_\varepsilon \cdot \mu\left(\bigcup_{j \in R_n(x)} B_n(y_j, \varepsilon)\right) \leq \chi_\varepsilon \cdot \mu(y, |\Sigma_n(g, y)| \geq \eta) \end{aligned}$$

But from (6) we can control the total measure of the set of bad n -preimages, which is thus smaller than ε' . Hence

$$\sum_{1 \leq k \leq r} \mu(B_{l_k}) + \Sigma(S, n) + \Sigma(T, n) \leq \chi_\varepsilon \cdot \varepsilon' \tag{13}$$

This procedure can be used for any collection of mutually disjoint borelian subsets from the collections $H(\ell, n, \varepsilon)$, $1 \leq \ell \leq d^n$, not only for S, T . Indeed by using Lemma 2 and the disjointness of sets from $H(\ell, n, \varepsilon)$ (hence also the mutual disjointness of the sets of their n -preimages) we see that the weights associated in (12) to any measure $\mu(B_j)$ (where B_j corresponds to a bad n -preimage) never add up to more than 1.

Then similarly as in (13), by employing the control on the total measure of bad n -preimages from (6), we can conclude that for $n > n(\eta)$:

$$\mu(\Lambda \setminus D_n(\beta, \eta)) \leq C_\varepsilon \cdot \frac{\varepsilon'}{\beta} \cdot \chi_\varepsilon = \tilde{C}_\varepsilon \cdot \frac{\varepsilon'}{\beta} \quad (14)$$

Therefore by using (9) and (14)

$$\begin{aligned} \int_{\Lambda} I_n(g, x) d\mu(x) &\leq 2\eta + \int_{\Lambda(\eta)} I_n(g, x) d\mu(x) \leq \\ &\leq 2\eta + \int_{D_n(\beta, \eta)} I_n(g, x) d\mu(x) + \int_{\Lambda \setminus D_n(\beta, \eta)} I_n(g, x) d\mu(x) \leq \\ &\leq 2\eta + 2\|g\|\beta + 2\eta + 2\|g\|\mu(\Lambda \setminus D_n(\beta, \eta)) \leq 4\eta + 2\|g\|\beta + 2\|g\|\tilde{C}_\varepsilon \frac{\varepsilon'}{\beta}, \end{aligned}$$

for $n > n(\eta)$. Recall however that we assumed before that $3\eta = \beta$. Assume that ε' is so small that $\tilde{C}_\varepsilon \cdot \frac{\varepsilon'}{3\eta} < \eta$. Then from the last displayed inequality, it follows that there exists a positive constant $C' = 4 + 8\|g\|$ so that:

$$\int_{\Lambda} I_n(g, x) d\mu(x) \leq C' \cdot \eta, \text{ for } n > n(\eta)$$

This shows in conclusion that

$$\int_{\Lambda} I_n(g, x) d\mu(x) \xrightarrow{n \rightarrow \infty} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R}).$$

□

Hence we proved the convergence in integral (with respect to $d\mu_n(x)$) of the measures μ_n^x from (1), towards the equilibrium measure μ_ϕ of ϕ , in the hyperbolic non-invertible case.

Corollary 1. *In the same setting as in Theorem 1, for any Holder potential ϕ , it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak convergence of measures*

$$\mu_{n_k}^z \xrightarrow{k \rightarrow \infty} \mu_\phi$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0 -almost all points $x \in \Lambda$,

$$\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{\sum_{i=0}^{n_k-1} \delta_{f^i y}}{\text{Card}(f^{-n_k}(x) \cap \Lambda)} \xrightarrow{k \rightarrow \infty} \mu_0, \text{ for a subsequence } (n_k)_k.$$

Proof. Let us fix $g \in \mathcal{C}(\Lambda, \mathbb{R})$. From the convergence in μ_ϕ -measure of the sequence of functions $z \rightarrow \mu_n^z(g), n \geq 1$ obtained from Theorem 1, it follows that there exists a borelian set $E(g)$ with $\mu_\phi(E(g)) = 1$ and a subsequence $(n_p)_p$ so that $\mu_{n_p}^z(g) \xrightarrow{p} \mu_\phi(g), z \in E(g)$.

Let us consider now a sequence of functions $(g_m)_m$ dense in $\mathcal{C}(\Lambda, \mathbb{R})$. By applying a diagonal sequence procedure we shall obtain then a subsequence $(n_k)_k$ so that $\mu_{n_k}^z(g) \xrightarrow{k} \mu_\phi(g), \forall z \in \bigcap_m E(g_m)$. We notice also that $\mu_\phi(\bigcap_m E(g_m)) = 1$, since $\mu_\phi(E(g_m)) = 1, m \geq 1$. But since any continuous function g can be approximated in the uniform norm by a function g_m , it will follow that

$\mu_{n_k}^z(g) \xrightarrow[k]{\rightarrow} \mu_\phi(g), \forall z \in E := \bigcap_m E(g_m)$. Therefore we obtain that $\mu_{n_k}^z \xrightarrow[k]{\rightarrow} \mu_\phi, z \in E$, i.e we have weak convergence of the measures $\mu_{n_k}^z, k \geq 1$, on a set of z having full μ_ϕ -measure in Λ .

□

The Theorem applies to **Anosov endomorphisms** in particular.

Corollary 2. *Assume that $f : M \rightarrow M$ is an Anosov endomorphism without critical points on a Riemannian manifold. Let also ϕ a Holder continuous potential on M and μ_ϕ the equilibrium measure of ϕ . Then*

$$\int_M | < \frac{1}{n} \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{e^{S_n \phi(y)}}{\sum_{z \in f^{-n}(x) \cap \Lambda} e^{S_n \phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \xrightarrow[n \rightarrow \infty]{} 0, \forall g \in \mathcal{C}(M, \mathbb{R})$$

We can compare these results to the usual SRB measure for the endomorphism f , defined as a measure μ having the property that for any measurable partition η of \hat{M} subordinate to the lifts of the local unstable manifolds, and for $\hat{\mu}$ almost all $\hat{x} \in \hat{M}$, the projection of the conditional measure of $\hat{\mu}$, namely $\pi_*(\hat{\mu}_{\hat{x}}^\eta)$ is absolutely continuous with respect to the induced Lebesgue measure on $W_{\hat{x}}^u$ ([4], [12]). In [12] it is shown that μ satisfies the SRB property for the Anosov endomorphism f if and only if $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} \xrightarrow[n]{\rightarrow} \mu$ for Lebesgue almost every $x \in M$.

Corollary 3. *Let $f : M \rightarrow M$ be an Anosov endomorphism, $\phi : \Lambda \rightarrow \mathbb{R}$ a Holder potential and assume that the equilibrium measure μ_ϕ is absolutely continuous with respect to the Lebesgue measure on M . Then the measure μ_ϕ with this property is unique, it is an SRB measure and it also satisfies an inverse SRB condition in the sense that there exists a set E of full Lebesgue measure in M and a sequence $(n_k)_k$ such that $\mu_{n_k}^z \xrightarrow[k]{\rightarrow} \mu_\phi, z \in E$.*

Proof. The proof follows immediately from Theorem 1, Corollary 2 and from the results of [4] and [12]. The potential ϕ for which μ_ϕ is SRB, can be taken in fact to be the unstable potential.

□

A classical example of Anosov endomorphism without critical points is a **toral hyperbolic endomorphism** $f_A : \mathbb{T}^m \rightarrow \mathbb{T}^m$, associated to an $m \times m$ integer valued matrix A whose eigenvalues λ_i all have absolute values different from 1, and whose determinant $\det(A)$ is not necessarily 1 in absolute value. Each point from \mathbb{T}^m has exactly $|\det A|$ preimages in \mathbb{T}^m . If we consider the potential $\phi \equiv 0$, then the equilibrium measure of ϕ is the Haar measure ω on \mathbb{T}^m which is also the measure of maximal entropy (its entropy is equal to $\sum_{\lambda_i, |\lambda_i| > 1} \log |\lambda_i|$, where each eigenvalue is taken with its multiplicity). In [17] it was proved the asymptotic distribution of periodic points towards ω ; here we prove the convergence towards ω , which is also the unique measure of maximal entropy, of the measures μ_n^x, n corresponding to the potential $\phi \equiv 0$, for ω -almost all points $x \in \mathbb{T}^m$. We thus obtain the existence of an *inverse SRB measure* in this case.

Moreover Theorem 1 applies also to smooth (say \mathcal{C}^2) perturbations $f_{A,\varepsilon}$, of hyperbolic toral endomorphisms f_A . Indeed they will also be hyperbolic on the m -dimensional torus \mathbb{T}^m and the

basic set considered is the whole \mathbb{T}^m . Also the non-invertible map $f_{A,\varepsilon}$ remains $|\det(A)|$ -to-1 on \mathbb{T}^m . We thus obtain the weighted distribution (with respect to a Holder potential ϕ on \mathbb{T}^m) of preimage sets of $f_{A,\varepsilon}$, with respect to the equilibrium measure μ_ϕ of ϕ , for **perturbations of hyperbolic toral endomorphisms**.

Anosov endomorphisms on **infranilmanifolds** represent a generalization of toral linear endomorphisms (see the Remark at the end of Section 1). Let us notice that our Theorem 1 applies to Anosov endomorphisms on infranilmanifolds which are not topologically conjugate to Anosov diffeomorphisms nor to expanding maps. Thus, besides Theorem 1, one cannot apply any of the previously known results for the distributions of preimages from the case of diffeomorphisms ([2]), or expanding endomorphisms ([13]).

Theorem 1 applies also to **hyperbolic basic sets of saddle type** for endomorphisms which *are not necessarily Anosov*, like the class of examples from [10], namely skew products with overlaps in their fibers $F : X \times V \rightarrow X \times V, F(x, y) = (f(x), h(x, y))$, where $f : X \rightarrow X$ is an expanding map on a compact metric space, while $h(x, \cdot) : V \rightarrow V$ (denoted also by h_x) is a contraction on an open convex set $V \subset \mathbb{R}^m$; h_x is assumed to depend continuously on $x \in X$. The basic set is in this case given by

$$\Lambda := \bigcup_{x \in X} \bigcap_{n=0}^{\infty} \bigcup_{z \in f^{-n}x} h_z^n(\bar{V}),$$

where $h_z^n := h_{f^{n-1}z} \circ \dots \circ h_z, n \geq 1, z \in X$. In [10] we studied the conditional measures of equilibrium states induced on fibers and their relation to the stable dimension of fibers. So from Theorem 1 we obtain the weighted distributions of preimages of the non-invertible map F over Λ , with respect to equilibrium measures of Holder potentials.

We can collect the above remarks in the following:

Corollary 4. *a) The conclusions of Corollary 2 hold in particular for toral hyperbolic endomorphisms and for smooth perturbations of these.*

b) The conclusions of Theorem 1 hold for the basic sets of hyperbolic skew products with overlaps in their fibers from [10], as well as for the attractors of the noninvertible horseshoes from [1].

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